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Research Article

APOS Framework Didactize Equivalence Linear Simultaneous Equations in Senior High School Mathematics

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Abstract
Many students followed rote-learned procedural rules without thinking about the meaning of quadratic equations, and the Chief examiners' reports attributed the trend to poor teaching methods. In certain instances, candidates cannot use even the conventional methods to factorize quadratics equations due to a lack of understanding of the zero-product property in graphing, factorizing, and completing the squares or quadratic formula. In the worst circumstances, the candidates failed to scaffold higher-order quadratic equations using the Conjugale and Equivalence Linear Simultaneous Equation methods. However, the action, process, object, and schema theory which came out of the constructivist learning theory and created by Dubinsky can be applied to teach mathematics. With the aid of the APOS framework, this study sought to digitize the Equivalence Linear Simultaneous Equations in senior high schools. In this mixed-method embedded design, there were 286 first-year students selected from one school. All the students received four phases of the APOS framework. The four phases were collected based on the Actions for Factorization, Processes for Quadratic Formula, Object for Conjugale, and Schema for ELSE method. The data, which was both qualitatively and quantitatively, was analyzed using IBM SPSS Statistics (Version 26) deterministic analytics software. The data collection covered four contact periods of a total duration of four hours. The results showed that students in the Actions and Processes were not statistically significant. However, the results were statistically significant at the Object and Schema phases. It was concluded that students' learning through the APOS framework improved their academic performance. The positive effect was the triumphant mental constructions involving encapsulation and interiorization in the conjugale and ELSE methods. It was, therefore, recommended that the framework be promoted to find solutions to many other complex mathematics problems.

Keywords: APOS Framework, Conjugale, Equivalence Linear Simultaneous Equations, Mathematics, Mixed-Method Embedded

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1. INTRODUCTION

Didis Kabar (2018) opines that quadratic equation research studies involve different foci, two of whom are geometric approaches to and historical perspectives on quadratic equations. The history of quadratic equations began with the pre-historic people who collected, grouped, counted, and added objects. The Egyptian era did rudiments of quadratic equations without finding solution sets. The Babylonians handled equations effectively by adding equal terms to both sides. The Greek civilization applied Pythagorean geometrical algebra and conic sections by completing the square but omitting the roots. The Arabs discovered three classical cases of quadratic trinomials, namely squares and roots equal to numbers, squared and roots equal to roots, and roots and numbers equal to squares (i.e. x^2, x and constant 'c' respectively). The Hindus, working independently of the Arabs, obtained the algebraic solution of $ax^2+bx+c=0$. The Hindu method divided the quadratic equation into three fundamental types,

$x^2 + ax = b$, $x^2 + b = ax$, and $x^2 = ax + b$ (Chorla, Clark & Tzanakis, 2022). In these three types, only the positive coefficients were admitted (negative quantities standing alone were still not accepted by the Arabic mathematicians) (Ali, Davis & Agyei, 2021; Rogers & Pope, 2015; Sönnerhed, 2021).

In Medieval Europe, the negative coefficients in equations reduced the diversity of cases. This led to the solution of the quadratic equations of the form $x^2 + px + q = 0$, where p and q are constants. In modern times, all second-degree equations could be written in the form $ax^2 + bx + c = 0$, where 'a', 'b', 'c' are constants, and x is a variable (Chorla, Clark & Tzanakis, 2022). Quadratic equations have always been in the syllabi of pre-tertiary levels of Mathematics teaching and learning. Didis Kabar (2023), Baybayon and Lapinid (2024), and O'Connor and Norton (2024) agreed that the conceptual meaning of quadratic equations is generally ignored in the teaching of quadratic equations. This compels students to follow rote-learned procedural rules without thinking about their meaning to solve quadratics (Makgakga, 2020; Tendere & Mutambara, 2020).

The Chief examiner's reports of the West African Examination Council (2023) attributed the poor performance in solving quadratic questions to poor methods. In one instance, candidates find it difficult to factorize the equations. This is due to inadequate local innovation added to the methods and techniques (Ali, 2021; Da, 2023). This defeats the aim of producing a mathematically functional workforce in society.

Again, candidates cannot use conventional methods to factorize quadratics equations. Thomas and Mahmud (2021) attributed the problem to a lack of prior knowledge from previous grades. Gözde and Kabar (2018) traced the problem to a lack of understanding of the variable concept, and confusing quadratic equations and linear equations.

Moreover, Makgakga (2023) argued that quadratic equations always have two solutions, roots, or zeros by using the zero-product property which states that if the product of two quantities equals zero, at least one of the quantities should be zero. This is the basis for applying the four conventional methods for solving quadratic equations using graphical, factorization, completing the square and quadratic formula. But it appears the conjugale and ELSE have long been discarded and disappeared.

In addition, Mutambara et al. (2020) discovered that mathematical understanding exists as procedural knowledge, conceptual knowledge, or both. Conceptual understanding involves content mastery where knowledge can be generated and established through many relations between existing and prior knowledge and transferred through reconstruction of procedures, a student who performs well in class may appear to have a basic understanding of quadratic function concepts, but in reality, they may not have a conceptual understanding of the concept. This was proved by videotaping two students working together on a problem and comparing their written work and oral interviews (Mutambara, 2020).

1.1 Theoretical Framework

This study applied the Action–Process–Object–Schema theory (APOS) to navigate from simple factorization to complex analysis in ELSE. The APOS theory states that mathematics teaching and learning should be based on helping students use the mental structures they already have and develop new, more powerful structures (Borji, et al., 2018). Dubinsky (1991) argues that the formation of concepts occurs first as an activity, followed by a process, supported by an object, and then ends at a schema. The actions and processes are operations on a previously established object and each action needs to be interiorized into a process and then encapsulated into an object before being acted upon by other actions/processes (Dubinsky, & McDonald, 2020). The emergence of APOS theory provided this lifelong narrative about mathematics learning in the realms of epistemology and constructivism (Tsafe, 2024).

Explained further by Listiawati and Juniati (2021), an action is a transformation of objects perceived by the individual as essentially external and requiring, either explicitly or from memory, step-by-step instructions on how to operate. When an action is repeated and the individual reflects upon it, they can make an internal mental construction called a process which the individual can think of as performing the same kind of action, but no longer with the need for external stimuli. An object is constructed from a process when the individual becomes aware of the process as a totality and realizes that transformations can act on it. A schema for a certain mathematical concept is an individual's collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept (Listiawati, & Juniati, 2021).

As applied in this study (see Figure 1), the factorization was first formed as an action, which was an externally directed transformation of previously conceived objects. A process in the quadratic formula was followed as the same factorization, the student had to imagine performing the corresponding factorization without having to execute each step explicitly. Now, the students become aware of a mental process and can construct transformations acting on the quadratic formula into a cognitive object from the conjugale. At this level, students could organize a coherent framework called schema, to mentally construct and deal with ELSE situations. In this last stage, the actions, processes, objects, and schemas have enabled students to successively mentally construct the interiorization, and encapsulation to build those mental constructions (Borji, et al., 2018).

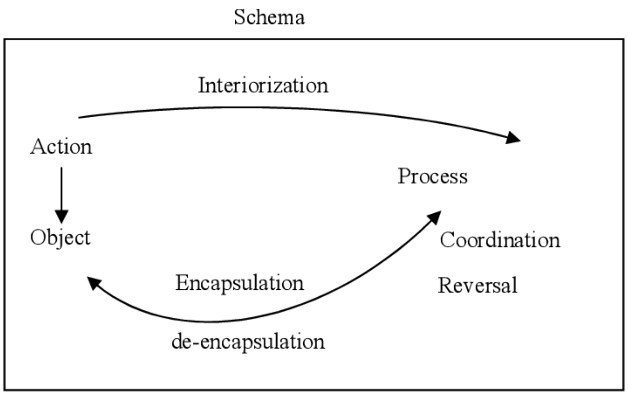

Figure 1. Conceptual Framework (adopted from Syamsuri & Marethi, 2018)

1.1.1. Action during the Quadratic Factorization

Factorizations fit into a sequence of actions as follows:

Given $ax^2 + bx = -c$

- $x^2 + (a + b)x + ab = 0$
- $x^2 + ax + bx + ab = 0$
- $x(x + a) + b(x + a)$
- $(x + a)(x + b) = 0$

Therefore, x = -a or x = -b (Clinch, 2018)

1.1.2. Process during the Quadratic Formula

The quadratic formula fits into process conception and is interiorized into a mental process as follows:

- Given $ax^2 + bx + c = 0$
- Divide through by 'a' obtain: $x^2 + \frac{bx}{a} + \frac{c}{a} = 0$
- Adding $\left(b/_{2a} \right)^2$ to both sides, we get $x^2 + \frac{bx}{a} + \left(\frac{bx}{a}\right)^2 + \frac{c}{a} = \left(\frac{bx}{a}\right)^2$

- Re-arranging terms gives $x^2+\frac{bx}{a}+\left(\frac{bx}{a}\right)^2=\left(\frac{bx}{a}\right)^2-\frac{c}{a}$, and hence, $\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$
- Taking square root on both sides gives $x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$
- Therefore $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ …………………………………… (1) (Mbewe & Nkhata, 2019)

1.1.3. Object during the Conjugale

The conjugale can be drawn by a perceived entity upon which actions and processes could be made:

Given $ax^2+bx=-c$

- Multiply equation (1) by $4a$: $4a^2x^2+4abx=-4ac$
- Add to b^2 both sides of equation (2): $4a^2x^2+4abx+b^2=b^2-4ac$
- The left-hand side of equation (3) is now a perfect square.
- $\Rightarrow(2ax+b)^2=b^2-4ac \Rightarrow 2ax+b=\pm\sqrt{b^2-4ac}$
- Let d be any number equivalent to $\pm 2ax+b$
- Implying that $d=\sqrt{b^2-4ac}$ ……………………………………………… (2)
- and hence $2ax+b=-d$ or $2ax+b=d$ …………………………………… (3)

1.1.4. Schema during the ELSE

The schema begins when a new idea is thematized and ends when it becomes a basis for a higher–level concept in the ELSE method. This cycle adds another layer to the system of conjugale:

Given $ax^2+bx=-c$

- If x_1 and x_2 are the roots of the quadratic equation $ax^2+bx+c=0$; then:
- $ax_1^2+bx_1+c=0$ ………………………………..………..………. (4)
- $ax_2^2+bx_2+c=0$ ………………………………………………...………. (5)
- Subtracting equation (5) from (4) we get;
- $ax_1^2-ax_2^2+bx_1-bx_2=0$ or $a\left(x_1^2-x_2^2\right)+b\left(x_1-x_2\right)=0$
- $a\left(x_1+x_2\right)\left(x_1-x_2\right)=-b\left(x_1-x_2\right)$ or $a\left(x_1+x_2\right)=-b$
- From the equation above, $b=-a\left(x_1+x_2\right)$. And so, putting it in equation (4) we have,
- $ax_1^2-a\left(x_1+x_2\right)x_1+c=0$ or $ax_1^2-ax_1^2-ax_1x_2+c=0$

- $-ax_1x_2 + c = 0$ or $ax_1x_2 = c$
- Squaring both sides of the equation (3), we get,
- $a^2(x_1 + x_2)^2 = (-b)^2$ or $a^2(x_1 + x_2) = b^2$
- Multiplying both sides of equation (4) by 'a' we get $a^2x_1x_2 = ac$
- Also, it is a property for real numbers that, $(x_1 - x_2)^2 = (x_1 + x_2) - 4x_1x_2$
- Multiplying both sides by a^2 gives,

$a^2(x_1 - x_2)^2 = a^2(x_1 + x_2)^2 - 4a^2x_1x_2$

- $a^2(x_1 - x_2)^2 = b^2 - 4ac$
- $a(x_1 - x_2) = \sqrt{b^2 - 4ac} = d$, where $b^2 - 4ac \succ 0$ or $ax_1 - ax_2 = d$
- From equation (3); $a(x_1 + x_2) = -b$ or $ax_1 + ax_2 = -b$
- Thus, the simultaneous linear equations;
- $ax_1 + ax_2 = -b$ …………………………………………………….. (6)
- $ax_1 - ax_2 = d$ ………………………………………………………. (7)
- This can be solved easily for x_1 and x_2 .

1.2 Purpose of the Study

The purpose of the study was to use the APOS framework to didactize solutions in quadratic equations

1.3 Research Questions/ Null Hypothesis

1. How does the APOS framework help to solve quadratic equations?
2. How do student solutions in the quadratic equations differ in the APOS framework?

Consequently, the researchers derived the following two hypotheses from the Research Question 2:

1. H_{01}: There are no statistically significant differences between the methods
2. H_{02}: There are no statistically significant differences between the programmes.

2. METHODS

2.1 Research Design

In Figure 2, the embedded mixed methods design was used to collect quantitative and qualitative data simultaneously, but the qualitative data was embedded within the quantitative data. This design was best used to focus on the quantitative data but still needed to understand how the qualitative data further explained it. The researchers used a pretest-posttest non-equivalent group design as all the students went through factorization, quadratic formula, conjugale, and ELSE methods (Ng & Chew, 2023).

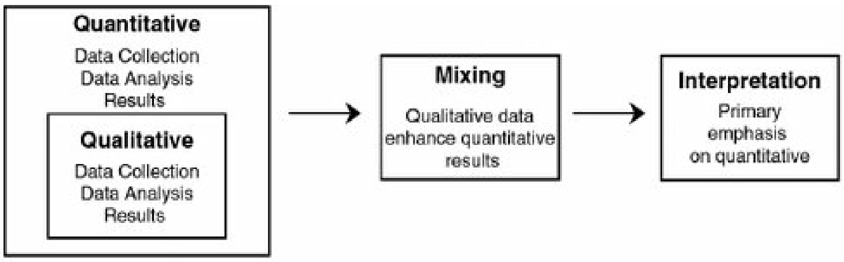

Figure 2. Embedded Mixed-Method Research Design (adopted from Creswell & Creswell, 2018)

The purpose was to assess the potential of students in navigating through the APOS framework to solve quadratic equations. In this design, tasks were planned and administered to the students using the APOS framework (Nagle, Martínez-Planell & Moore-Russo, 2019). It covered four contact periods of a total duration of four hours in the factorization, Quadratic Formula, Conjugale, and ELSE methods. The Factorization, Quadratic Formula, and Conjugale served as control while ELSE represented the experimental treatment. The achievement test's mean scores were then compared (Listiawati, & Juniati, 2021).

2.2 Participants

One is bound to encounter challenges when the universe to be sampled is not precisely defined. In this study, a population is the entire aggregation of cases that meet a designated set of criteria. The target population consists of 600 students in form one in the Suhum Senior High/Technical School in the Municipality of Ghana. The accessible population was 155 first-year students pursuing mathematics and physics in the school.

The choice of the sampling technique was guided by Nga et al. (2023) as follows:

1. In many cases a complete coverage of the population is not possible.
2. Complete coverage did not offer a substantial advantage over a sample survey.
3. Studies based on samples required less time and produced quick answers.
4. It was less demanding since it required a small portion of the target population.
5. It was considered more economical since it contained fewer elements of the population
6. It required fewer printed materials and a reduced general cost.
7. The samples offered more detailed information and a high degree of accuracy.
8. It required convenience and more proximity of schools to the researchers.

Therefore, the stratified technique was used to select two groups of students. This technique involved dividing the population into many homogenous groups or strata. Each group contains subjects had similar characteristics. A sample was then drawn from each group. The eight classes selected had a total population of 600. The sample size was 286 students consisting of 160 girls and 126 boys who took part in the experiment (Borji, et al., 2018).

In the Schools, the Assistant Headmaster in consultation with the mathematics teachers selected all the students who have learning difficulties in Mathematics. A round of ballots determined a method for inclusion and exclusion criteria (Nga et al., 2023). The eight sampled classes were Business, General Arts, General Science, Visual Arts, Building and Construction, and Home Economics. In addition to Mathematics and Physics who were added independently because of their central roles in quadratic equations.

2.3 Research Instruments

The pilot of the instruments was undertaken at Presbyterian Senior High School in Suhum during the first semester. A formal introduction was made to the head of the selected school three weeks before the start of the data collection exercise. With the permission of the school authority, the Head of the Mathematics Department assigned the six classes. All students in these six classes had done concepts related to quadratic equations (Nga et al., 2023).

There were two instructional periods of one hour each duration for an achievement test based on the lessons. All the lessons followed the school's normal timetable. Exercises for class discussions and practice were selected from the previous year's group. However, care was taken to exclude the questions previously selected from the textbook for the achievement test from the discussions. We undertook all the teaching experiments using the following criteria (Dubinsky, & McDonald, 2020; Manzouri, 2024).

2.4 Procedures

The items followed the same order as they appeared in the textbook. Students answered the questions on the question paper because space was created on it. The test was by nature one for achievement that was crafted to elicit both knowledge (concept, recognition, recall) and comprehension of the subject matter of quadratic equations. This was made amply clear to the students throughout the teaching experiment (Brannen, 2018; Nga et al., 2023; Schoonenboom & Johnson, 2017).

All the common topics among the classes constitute pre-requisite skills for the solutions of quadratic equations. These items were drawn from areas such as factorization of algebraic expression, the solution to linear equations, real number system, and numeration system. The items gave the researchers a fair idea of the state of learning readiness for the solution set of quadratic equations ((Dubinsky, & McDonald, 2020).

A marking scheme was drawn for each test, such that there was no differential based on the treatment. So, no one treatment gained an undue advantage over the other. The posttest items followed the same pattern as the pretest. Each item of the posttest was awarded ten (10) marks. Every logical Mathematics step was awarded a mark including the final answer (Listiawati, & Juniati, 2021).

2.5 Data Analysis

The researchers first used content analysis of the four methods to display the life experiences of the content (Pinilla, 2024). The researchers used the t-test, Analysis of variance (ANOVA), and Multivariant Analysis of Variance (MANOVA). The test statistics assumed a random sampling, homogeneity of variance, normal distribution, and equal population covariance for the MANOVA. The tools were considered valuable since the subjects were not randomly assigned to the treatment groups. This increased the statistical power by controlling variability due to the effects of extraneous variables and reduced bias in statistical analysis (Mukavhi, Brijlall & Abraham, 2021).

The t-test for both independence and paired samples was used to analyze the scores. The t-test for dependent (paired) samples was used to analyze the mean scores in the four methods for solving quadratic equations (Tiengyoo, Sotaro, & Thaithae, 2024). The ANOVA procedure was designed to measure the spread in the data and portion, the total amount of variance presented for all the sample data into two components, one corresponding to what happens between the different sets of data and the other corresponding to what happens within each set of data. If there was a little variance within each sample group and there was relatively little difference variance within each sample group and there was relatively little difference between sample units, then we cannot conclude that the population means were different (Nga et al., 2023).

The MANOVA was used to analyze the test scores of all four methods in the APOS framework. The entire tests were pegged at a 0.05 level of significance with research hypotheses and acceptance criteria (Mukavhi, Brijlall & Abraham, 2021).

2.6 Validity and Reliability

In this study, content and construct validities were most paramount. The content validity was related to how adequately the content of the items of the instruments (tests) and the responses to the test sampled the domain about which inferences were made. Content validity was built into the test from the outset. Thus, at the planning stage, a test specification table was drawn on the entire area of the methods of solving quadratic equations (Ng & Chew, 2023).

Also, the test items were constructed within the prescribed quadratic equations of the mathematics syllabus. Similar questions were taken from international examinations' past questions. The items of each test were given to some experienced Mathematics teachers for scrutiny. After which renowned mathematics education researchers went through it for the final scrutiny. These items were then validated following the comments from experienced teachers and researchers (Ng & Chew, 2023).

In this study, the reliability coefficients of the pretest were calculated using Kuder-Richardson's reliability (K20), and that of the posttest was calculated using the Cronbach Alpha. The pretest and the posttest recorded 0.86 and 0.80 respectively (Tiengyoo et al. 2024).

3. RESULTS

Research Question 1: How does the APOS framework help to solve quadratic equations?

Four themes emerged from this research question. There was the action in factorization, the process in quadratic formula, the object in conjugale, and the schema in ELSE methods.

Theme 1: Action during the Quadratic Factorization

Task 1: Find the roots of the quadratic equation $x^2 + 4x – 5 = 0$ by the method of completing the square.

Script 1:

Given $x^2 + 4x – 5 = 0$, where b = 4, c = -5

- $(x + b/2)^2 = -(c – b^2/4)$
- So, $[x + (4/2)]^2 = -[-5 – (4^2/4)]$ or $(x + 2)^2 = 5 + 4$
- $\Rightarrow (x + 2)^2 = 9$ or $\Rightarrow (x + 2) = \pm\sqrt{9}$ or $\Rightarrow (x + 2) = \pm 3$
- $\Rightarrow x + 2 = 3, x + 2 = -3$ or $\Rightarrow x = 1 , -5$

Therefore, the roots of the given equation are 1 and -5.

Task 2: Solve $x^2+7x+12=0$.

Script 2:

The quadratic will be in the form $(x+a)(x+b)=0$.

Given $x^2+7x+12=0$.

- Find two numbers with a product of 12 and a sum of 7.
- $3\times4=12$, and $3+4=7$, so a and b are equal to 3 and 4. This gives $(x+3)(x+4)=0$
- The product of x+3 and x+4 is 0, so x+3 = or 0 x+4 = 0, or both.
- $x+3 = 0−3−3x = −3$ or $x+4 = 0−4−4x = −4$ or $x = −3$ or $x = -4$

Theme 2: Process during the Quadratic Formula

Task 3: Solve $x^2 - 3x - 4 = 0$ using the quadratic formula.

Script 3:

Given a = 1, b = -3, and c = -4.

- $x = [-b \pm \sqrt{(b2 - 4ac)}]/2a = [-(-3) \pm \sqrt{((-3)2 - 4(1)(-4))}]/2(1)$
- $= [3 \pm \sqrt{25}]/2 = = [3 \pm 5]/2 = = (3 + 5)/2$ or $(3 - 5)/2$

- = 8/2 or -2/2 = = 4 or -1 are the roots.

Task 4. Solve for $4x^2+3x-5=0$

Script 4:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Given $4x^2+3x-5=0$

- Let $a=4$, $b=3$ and $c=-5$:
- $x=\frac{-3\pm\sqrt{(3)^2-4(4)(-5)}}{2(4)}$ = $x=\frac{-3\pm\sqrt{(9+80)}}{8}$
- $x=\frac{-3\pm\sqrt{89}}{8}$ $x=\frac{-3\pm9.433981}{8}$
- $x=\frac{-12.433981}{8}$ or $x=\frac{6.433981}{8}$

Therefore, $x=-1.554$ or $x=0.804$ (3 decimal places)

Theme 3: Object during the Conjugale

Task 5. Find the truth set of the equation $x^2-15x-100=0$

Script 5:

Given $x^2-15x-100=0$

- Compare $ax^2+bx+c=0$
- $a=1$, $b=-15$ and $c=-100$
- Putting these values into $d=\sqrt{b^2-4ac}$
- $d=\sqrt{(-15)^2-4(1)(-100)}$, $d=\sqrt{225+400}$, $d=\sqrt{625}$, $d=25$
- The two linear equations required can therefore be formed as,
- $2x-15=-25$………………………..……………………… (1)
- $2x-15=25$…………………………………………………… (2)
- From equation (1):
- $2x-15=-25$ or $2x=-25+15$ or $2x=-10$ or $\frac{2x}{2}=-\frac{10}{2}$ or $x=-5$
- $2x-15=25$ or $2x=25+15$ or $2x=40$ or $\frac{2x}{2}=\frac{40}{2}$ or $x=20$

Hence the truth set of the equation is $\{x:x=-5,20\}$.

Task 6. Find the truth set of the equation $x^2 - 5x + 6 = 0$

Script 6:

Given $x^2 - 5x + 6 = 0$,

- Compare $ax^2 + bx + c$, $a = 1$, $b = -5$ and $c = 6$
- Putting these values into
- $d = \sqrt{b^2 - 4ac}$ or $d = \sqrt{(-5)^2 - 4(1)(6)}$ or $d = \sqrt{25 - 24}$ or $d = \sqrt{1}$ or $d = 1$
- The two linear equations required can therefore be formed as,
- $2x - 5 = -1$ ………………..…………………. (1)
- $2x - 5 = 1$ …………………………………. (2)
- From equation (1) and (2), $x = 2$ and $x = 3$ respectively.

Hence the truth set of the equation is $\{x : x = 2, 3\}$.

Theme 4: Schema during the ELSE

Task 7. Solve the equation $x^2 + 2x - 15 = 0$.

Script 7:

Given $x^2 + 2x - 15 = 0$.

- Compared to $ax^2 + bx + c = 0$, $a = 1, b = 2, c = -15$
- $d = \sqrt{b^2 - 4ac} = \sqrt{(2)^2 - 4(1)(-15)} = \sqrt{4 + 60} = \sqrt{64} = 8$
- Now let x_1 and x_2 represent the roots of the equation, then,
- $x_1 + x_2 = -2$ ……………………… (1)
- $x_1 - x_2 = 8$ ………………...……….. (2)
- Adding (1) and (2) we get, and $2x_1 = 6$ or $x_1 = 3$
- Putting the value of in equation (1) we get, $3 + x_2 = -2$ or $x_2 = -2 - 3$ or $x_2 = -5$

Therefore, the solution is $x = -5$ or $x = 3$

Task 8. Solve the equation $x^2 + 5x + 6 = 0$

Script 8:

Given $x^2 + 5x + 6 = 0$

- Compare to $ax^2 + bx + c = 0$
- $a = 1, b = 5, c = 6$

- $d=\sqrt{b^2-4ac}$ $d=\sqrt{(5)^2-4(1)(6)}$ $d=\sqrt{25-24}$ $d=\sqrt{1}$ $d=1$
- Now let x_1 and x_2 represent the roots of the equation, then
- $x_1+x_2=-5$ ………………………… (1)
- $x_1-x_2=1$ ………………...……….. (2)
- Adding (1) and (2) we get;
- $2x_1=-4$ or $\frac{2x_1}{2}=-\frac{4}{2}$ or $x_1=-2$
- Putting the value of x_1 in the equation (2) we get,
- $-2-x_2=1$ or $x_2=-2-1$ or $x_2=-3$

Therefore, the solution is -2 or -3

Research Question 2: How do student solutions in the quadratic equations differ in the APOS framework?

H1: There are no significant differences between the quadratic formula and factorization.

Table 1. Independent T-Test Quadratic Formula And Factorization

Method	Mean	SD	Mean Difference	t	Sig. (2-tailed)
Quadratic	79.72	15.01			
			4.48	1.66	0.99
Factorization	75.24	16.17			

An Independent t-test was conducted to determine whether significant differences exist between the mean scores of respondents with the use of quadratic formula and factorization in solving quadratic equations in Table 1. It was revealed that there was no statistically significant difference between the mean scores of quadratic formula (M=79.72, SD=15.01) and factorization (M=75.24, SD=16.17) at $p<0.05$ alpha level in solving a quadratic equation and the null hypothesis is accepted. Even though the mean score of the quartic formula was higher than factorization, it could be concluded that respondents' understanding of the use of these methods in solving quadratic equations was not different.

H2: There is no significant difference between the quadratic formula and ELSE

Table2. Independent T-test of Quadratic Formula and ELSE Methods

Method	Mean	SD	Mean Difference	t	Sig. (2-tailed)
Quadratic	77.62	17.02			
			11.32	4.02	0.000**
ELSE	68.40	17.34			

An Independent t-test was computed to determine whether significant differences exist between the mean scores of respondents with the use of quadratic formula and ELSE in solving quadratic equations in Table 2. It was revealed that there was a statistically significant difference between the mean scores of quadratic formula (M=79.72, SD=15.01) and ELSE (M=68.40, SD=17.34) at $p<0.05$ alpha level in solving a quadratic equation and that the null hypothesis is rejected. The implication is that respondents understood the quadratic formula method better in solving quadratic equations than the use of the ELSE method. This provided a basis for the process to proceed to the object.

H$_3$: There is no significant difference between the quadratic formula and conjugale.

Table 3. Independent-test of Quadratic Formula and Conjugale methods

Method	Mean	SD	Mean Difference	t	Sig. (2-tailed)
Quadratic	79.72	15.01			
			4.48	1.66	0.99
Conjugale	79.41	16.63			

An Independent t-test was conducted to determine whether significant differences exist between the mean scores of respondents with the use of the quadratic formula method and conjugale method in solving quadratic equations in Table 13. It was found out there was no significant difference between the mean scores of the quadratic formula method (M=79.72, SD=15.01) and conjugale method (M=79.41, SD=16.63) at $p<0.05$ alpha level in solving a quadratic equation and that the null hypothesis is accepted. Even though the mean score of the quadratic formula was higher than the conjugale method, it could be concluded that respondents understanding of the use of these methods in solving the quadratic equations is not different.

H$_4$: There are no significant differences between Factorization and ELSE

Table 4. Independent test of Factorization and ELSE methods

Method	Mean	SD	Mean Difference	t	Sig. (2-tailed)
Factorization	75.24	16.17			
			6.84	2.24	0.017*
ELSE	68.40	17.34			

Independent t-test was conducted to determine whether significant differences exist between the mean scores of respondents with the use of factorization method and ELSE method in solving quadratic equations in Table 14. It was found that there was a statistically significant difference between the mean scores of the factorization method (M=68.40, SD=17.34) at $p<0.05$ alpha level in solving quadratic equations and the null hypothesis was rejected. The implication is that respondents understood the factorization method better in solving quadratic equations than the use of the ELSE method.

H$_5$: There is no significant difference between the mean scores of Conjugale and ELSE

Table 5. Independent-test of ELSE and Conjugale methods

Method	Mean	SD	Mean Difference	t	Sig. (2-tailed)
ELSE	68.64	17.36			
			-10.77	-3.89	0.000***
Conjugale	79.41	16.36			

An Independent t-test was conducted to determine whether significant differences exist between the mean scores of respondents with the use of the ELSE method and Conjugale method in solving quadratic equations in Table 5. It was found that there was a statistically significant difference between the mean score of the ELSE method (M=68.64, SD=17.36) and the Conjugale method (M=79.41, SD=16.36) at $p<0.05$ alpha level in solving a quadratic equation and that the null hypothesis is rejected. The implication is that the respondents understood the Conjugale method better than in solving quadratic equations with the use of the ELSE method.

H_6: There are no significant differences between the mean scores of the methods

Table 6. Comparison of Four Methods of Solving Quadratic Equations

I-Method	J-Method	Mean Difference		
		(I-J)	Std. Error	Sig.
Quadratic	Factorization	4.48	2.81	.111
ELSE		6.48	2.74	.013*
Conjugale	Factorization	4.17	2.65	.116
Quadratic	ELSE	11.32	2.81	.000**
Quadratic	Conjugale	0.31	2.72	.0909
Conjugale	ELSE	11.01	2.66	.000**

In Table 6, the MANOVA was computed to test whether a statistically significant difference exists among the mean scores of the four methods in solving quadratic equations. Levene's test of the homogeneity of variance was also computed to determine the appropriate post hoc multiple comparisons to be used to determine where the significant differences existed among the four methods because the F-test showed significant differences. The result showed that the variances that existed among the means of the four methods were statistically not significant at $p<0.05$ alpha level. This implies that equal variance is assumed among the four methods. Since equal variance was assumed, the Least Significant Difference (LSD) was chosen at the appropriate post hoc multiple comparison technique for the comparison of the mean differences among the four methods.

It was found that the quadratic formula method (M=79.72, SD=15.01) was significantly different from the ELSE method (M=68.40, SD= 17.34). However, it was not significantly different factorization method (M=75.24, SD= 16.17) and conjugale method (M=79.41, SD=16.36). Factorization method (M=75.24, SD=16.17) was also significantly different from ELSE method (M=68.40, SD=17.34) but was not significantly different from quadratic formula (M=79.72, SD=15.01) and conjugale (M=79.41, SD=16.36). The conjugale method (M=79.41, SD=16.36) was significantly different from the ELSE method (M=68.40, SD=17.34).

Table 7. Pairwise Comparisons of the Four Methods

Methods	ELSE	Quadratic	Conjugale	Factorization	Total
Quadratic	4		2	2	8
Factorization	3	2	2		7
ELSE		0	0	1	1
Conjugale	4	2		1	8

In Table 7, pairwise comparisons of the four methods were determined. Also, by awarding four points to a method when it is significantly higher at $p<0.01$ and zero points when it is significantly lower. Three points to a method when it is significantly higher at $p<0.05$ and one point when it is significantly lower. Two points have been awarded to each method when there are significant differences between them Table 18 displays the results.

In Table 7, the quadratic formula and conjugale had eight points each. There are totals of eight, seven and one points respectively for quadratic formula and conjugale, factorization, and ELSE. These results show that the students can learn best to solve quadratic equations by quadratic formula and conjugale, factorization, and ELSE in that order. These occurred between the following pairs of treatment groups:

- The quadratic formula and ELSE groups in favor of the quadratic formula group.
- The factorization and ELSE groups are in favour of the factorization group.

- The conjugale and ELSE groups in favour of the conjugale group.

4. DISCUSSION

The following were the Research Questions promulgated to guide the study:

a. How does the APOS framework help to solve quadratic equations?
b. How do student solutions in the quadratic equations differ in the APOS framework?

Makgakga (2023) argued that learners lack conceptual understanding in using only Factorization. The students successfully navigated the solutions of the quadratic equations using the APOS framework to tackle factorization with Actions, quadratic formula with Processes, Conjugale with Objects, and ELSE with Schema. In about seven different comparisons, the results showed that the students performed better after ascending to the ELSE with the Schemas.

The two transcripts in the Action of Factorization showed that transformation was first conceived as a reaction to stimuli. It then required specific instructions and the need to perform each step of the transformation explicitly (Ali, Davis & Agyei, 2021); Syamsuri & Marethi, 2018). Without the next steps, many learners use Factorization as rote memorization because they regard the two to be the same (Mutambara et al., 2020). The transcripts in the Process incorporated the actions of the Factorization into algorithms or procedures to help abstract the main characteristics of the quadratic formula, take control of them, use them flexibly, and interiorize them into mental processes to transition into the conjugale phase. The two transcripts in the schema aided the students in operating objects at a higher–level of the conjugale phase. All these enabled the schema of the students to understand and make sense of the perceived problem situation (Syamsuri & Marethi, 2018).

The significances of the transcripts show that complex quadratic equations require well-structured framework that can take each level of the concept at a time. The APOS framework has far-reaching impacts beyond factorization, quadratic formula, conjugale, and ELSE methods. The framework has created alternative pathways to scaffolding cubic, quartic, and other higher-order polynomials whose solutions are more complicated. This is one of the cogent ways of using quadratic equations to calculate areas of an enclosed space, speed of moving vehicles, profit and loss of business enterprises, bridging rivers and streams, and curving pieces of equipment for tessellations (Borji, Alamolhodaei & Radmehr, 2018).

It was found that the Factorization and quadratic formula methods were not statistically significantly different. However, these methods were significantly different from the conjugale and ELSE methods. Even the conjugale method was significantly different from the ELSE method. The significant differences could only be attributed to the manner the Framework redefined actions and interiorized the processes. The positive effect was the triumphant mental constructions involving encapsulation and interiorization in the conjugale and ELSE methods (Borji et al., 2018). The ultimate goal was the ability of the students to navigate their way from the Factorization to the ELSE. In this way, their curiosity and motivation were evoked and sustained for lifelong learning (Nga et al., 2022).

The APOS framework contributed immensely to the performance in each stage of the student's performance due to the efficacy of the theory in addressing cognition-related problems (Tsafe, 2024). In the patterns of movement from the Factorizations to the ELSE, the framework epitomizes its resilience and reliance in the milieu. Even though the framework is not a conventional learning theory, it does provide systematic, sequential, and step-by-step progress from the simplest to the complex quadratic constructs (Nagle, Martínez-Planell & Moore-Russo, 2019).

5. CONCLUSION AND RECOMMENDATIONS

The researchers used the APOS framework on Actions for Factorization, Processes for Quadratic Formula, Objects for Conjugale, and Schema for ELSE method to collect data from 286 students on quadratic equations. The data collection processes covered four contact periods of a total duration of four hours to answer the following research questions:

a. How does the APOS framework help to solve quadratic equations?

b. How do student solutions in the quadratic equations differ in the APOS framework?

The findings of the transcripts showed that the easiest way to navigate the framework was to use the action on factorization, processes on quadratic formula, object on conjugale, and schema on ELSE. The encapsulated processes propel wonder novelty in the solutions and applications of the quadratic equations. This yields curiosity and motivation for sustained lifelong learning.

Again, there were NO statistically significant differences between the mean scores on the Factorization and Quadratic Formula. This gave room to compare the methods conjugale and ELSE methods. However, there were statistically significant differences between the three methods and ELSE. All students' performance in ELSE was significantly lower than that of the factorization, quadratic formula, and conjugale. This means the actions were really interiorized into the processes and then encapsulated into objects before being acted upon by other actions and processes.

The following recommendations were made for policy, theory and practice:

The APOS framework stipulates teaching and learning of mathematics to use mental structures that they already have and to develop new more powerful structures, for handling more and more advanced mathematics. It was, therefore, recommended that factorization and quadratic formulas be strengthened with Conjugale and ELSE methods. This would surely give guidance and priority to replacing factorization and quadratic formulas in the curriculum.

The performance of the students was statistically significance and facilitated by the APOS framework would develop their skills to solve problems in daily life. It was, therefore, recommended that the mathematics curriculum be redesigned to encourage the use of the APOS framework.

6. LIMITATIONS OF THE STUDY

Despite the success of the applications of the APOS framework on quadratic equations, the researchers acknowledged that the sample size of 286 was drawn from one high school and may affect the generality of the findings. The data was also affected by our inability to identify students who already know the ELSE method and to properly confound these students in the research.

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