Journal of Research in Education and Pedagogy
ISSN 3047-7131 (online) & 3047-6410 (print)
May 2025, Vol. 2, No. 2, pp. 271–283

Scientia Publica Media

Research Article

Derivative Classes in Secondary Education

Circe Mary Silva da Silva[1]

Mathematics Education Program of the Federal University of Pelotas, Pelotas, Brazil

Abstract
The aim of this study was to identify how an experienced teacher introduces the concept of derivative in secondary education in Lisbon (Portugal) and to verify whether commognition, as proposed by Anna Sfard, is applicable in this context. The qualitative case study approach was adopted, data were collected in a public secondary school classroom, using observation techniques. The analysis focused on identifying the teacher's methodological approach, based on Sfard's four categories: Word Use; Visual mediators; Endorsed Narrative and Routine. However, routines were further subdivided into two subcategories: classroom routines and mathematical routines. Six episodes illustrating mathematical routines were analyzed in detail. The findings showed that symbolic, graphic and gestural visual mediators were consistently present in all lessons. The endorsed narrative was constructed through stated definitions and theorems that are demonstrated, and are consistently present in all lessons. The concept of derivative was constructed from the concept of average rate of change, followed by the notions of approximation, limit and finally derivative at a point. The approach to the concept of derivative was formalized, with some appeal to intuition. The study concludes that traces of commognition, as proposed by Sfard, are observed classes. Given the limited research on the teaching of derivatives at both the secondary and higher education levels, this study contributes valuable insights into how this fundamental concept of Differential Calculus is taught in secondary education.

Keywords: Differential Calculus, Derivative Function, Portugal, Teaching

Correspondence
Circe Mary Silva da Silva
cmdynnikov@gmail.com

Received
January 12, 2025
Accepted
April 16, 2025
Published
May 2, 2025

Citation: Silva, C. M. S. (2025). Derivative classes in secondary education. *Journal of Research in Education and Pedagogy, 2*(2), 271–283.

DOI: 10.70232/jrep.v2i2.67

1. INTRODUCTION

> One of the most important tasks for the academic community is to work with educators to develop codified representations of the pedagogical wisdom acquired through the practice of competent teachers (Shulman, 2014, p. 205 and 211).

I agree with Shulman (2014, p. 207) when he states that "pedagogical content knowledge is most likely the category that best distinguishes the understanding of a content expert from that of a pedagogue." Furthermore, the wisdom of practice, which can be observed in a more experienced teacher, is that which competent teachers master. In an article in partnership with Judith Shulman, the authors present the characteristics in the development of a competent teacher: vision, motivation, comprehension, practice, reflection, and community (Shulman & Shulman, 2016). It is not enough for a capable teacher to have vision, motivation, understanding, and practice; he or she must learn and reflect from his or her practice, listen to his or her students, and communicate with them in mathematical discourse.

Since the 1980s, researchers such as Tall and Vinner (1981), Tall (1991), Sierpinska (1985), Artigue (1998), Cornu (1991), Artigue; Viennott and Menigaux (2000), among others, have been researching the learning of the basic concepts of differential and integral calculus, such as limits, derivatives and integrals, mainly motivated by the cognitive difficulties that students manifest when faced with these new concepts. For Artigue (1998, p. 51): "Didactic research clearly shows that it is not easy for students to enter the conceptual field of Analysis, when this is not reduced to its algebraic part but rather to develop modes of thought and techniques that are currently based on it".

However, research on how teachers teach such concepts is less frequent. Nseampa and Gonzáles Martín (2022) found that there is little research on how derivatives are taught in classrooms, and the strategies used by mathematics teachers to teach this concept are also almost unknown. Bressoud et al. (2016) highlighted that investigation on teachers' practices in calculus are typically based on questionnaire and interview Analyses. However, Parker's research (2013, p. 248) was carried out with observations of classes of three instructors, concluding that "instructors' expertise blinds them to the difficulties students have in understanding what seems obvious. This indicates a disconnect between the endorsed narrative of the teacher and the students' abilities to comprehend what the teacher is saying". The research of Code et al. (2014, p. 26) conducted a comparative study in classrooms aimed to investigate the potential that a teaching model with interactive engagement; "The results revealed that students in the higher engagement classroom were more successful in connecting the procedures to new ideas". Mkhastshwa (2024, p. 9), in turn, researched the best practices for teaching the concept of derivative. It concluded that, there are several teaching strategies that are effective, such as "form of worksheets for students to explore the relationship between the concept of average rate of change and the derivative, creating opportunities to work with the concept of the derivative both in class and outside the class, using group problem solving strategies to help students master the concept of the derivative".

Park (2015), in her article *Is a derivative a function? If so, how do we teach it?* obtained her data in an investigation based on observations of Calculus I classes at Midwestern University (USA). She identified how three instructors approached the derivative at a point as a specific object and the derivative as a function on an interval. The author considered the components of the definition of a derivative, namely: function, difference quotient, and limit. While Park (2015) focused her research on teaching the derivative in a university context, my interest was to understand how the teaching of the concept of derivative is approached in secondary education in Portugal. Park's article provided me the theoretical inspiration to analyze the empirical data in another educational and geographic context – Portugal, in 2024. The objectives guiding the research were: to identify how an experienced teacher teaches derivatives in a secondary school in Lisbon city and to verify whether, in this teaching, *commognition* occurs, in the sense proposed by Sfard (2008a and 2008b).

2. THEORETICAL FRAMEWORK

The main theoretical basis of this research is based on the ideas of Sfard (2008a, 2008b). This researcher built her theory inspired by the ideas of Vygotsky and Wittgenstein and by empirical research on the teaching and learning of mathematics that she developed over many years.

For Sfard (2008b, p. 296), "commognition is a term that encompasses thinking (individual cognition) and (interpersonal) communicating; as a combination of the words - communication and cognition - , it stresses the fact that these two processes are different (intrapersonal and interpersonal) manifestations of the same phenomenon". She herself complements the meaning of the adjective *commognitive*: "To stress this fact, I propose to combine the terms cognitive and communicational into the new adjective commognitive" (Sfard, 2008, p. 83).

The words *commognition* and *commognitive* were thus defined by Sfard: "In the case of thinking, this general claim was translated into the statement that cognitive processes are individualized versions of interpersonal communication." (Sfard, 2008b, p.123). Therefore, when considering that thinking is an individualized version of interpersonal communication, it understands that everything that is created "is a product of collective action" (Sfard, 2008a, p. 432). For Sfard and Kieran, "Students' thinking is only understood in the context of demands and patterns of the overall communicative activity of which it is an inseparable part" (Sfard & Kieran, 2001, p. 47). Sfard (2008b, p. 296) defines communication as: "Communication a collectively performed patterned activity in which action *A* of an individual is followed by action B of another individual so that (1) *A* belongs to a certain well- defined repertoire of actions known as communicational, and (2) action *B* belongs to a repertoire of re-actions that fit *A*, that is, actions recurrently observed in conjunction with *A*".

Communication will be effective "if it fulfills its communicative purpose, that is, the different utterances of the interlocutors evoke responses that are in tune with the speakers' meta-discursive expectations" (Sfard & Kieran, 2001, p. 49). The word discourse is "used to denote any specific instance of communication, whether diachronic or synchronous, whether with others or with oneself, whether predominantly verbal or with the help of some other symbolic system" (Sfard & Kieran, 2001, p. 47). In 2008, Sfard provided the following definition of discourse: "special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions; every discourse defines its own community of discourse; discourses in language are distinguishable by their vocabularies, visual mediators, routines, and endorsed narratives" (Sfard, 2008b, p. 297).

In this work, the authors sought to detect "signs of communication violation" (Sfard & Kieran, 2001, p. 47). They stated that, "Communication cannot be regarded as effective unless, at any given moment, all the participants seem to know what objects they are talking about and feel confident that all the parties involved are referring to the same things when using the same words." (Sfard & Kieran, 2001, p. 51). Although she constructs a general theory, when referring to discourse, Sfard mainly targets mathematical discourse. One of the theses she defends is that the changes that occur in human practices are the result of transformations in *commognition*. Discourses are processes and not static entities and are constantly recreated in an intricate game of individualizations and communications. Mathematical discourse is highly abstract (Sfard, 2008b). She explains that there are rules and meta-rules in discourse: rules at the object level are narratives about regularities in the behavior of discourse objects, while meta-rules define patterns in the activity of the speaker who tries to substantiate narratives. For example, a rule in calculus can be exemplified when it is said that the derivative of the constant function is zero. Meta-rules will be exemplified in the narratives in the text that follows.

To characterize discourses, Sfard (2008b) presents categories that are summarized in Table 1.

Table 1. Characteristics of Mathematical Discourse in Sfard's *Commognition* Approach

Categories	Descriptions
Word Use	One of the distinctive characteristics of discourses is the keywords they use. In mathematics, these are mainly, although not exclusively, the words that signify quantities and shapes.
Visual mediators	Are visible objects that are operated upon as a part of the process of communication. In general, they involve symbolic artifacts.
Narrative	Is any sequence of utterances framed as a description of objects of relations between objects, and is subject to endorsement or rejection, that is, to being labeled as "true" or "false".
Routine	set of metarules defining a discursive pattern that repeats itself in certain types of situations; this set can be divided into three subsets: applicability conditions, routine course of action (or routine procedure), and closing conditions (or closure).

Source: Sfard (2008b)

Sfard (2008b) considers that the words used in schools and academies dictate their more disciplined uses, different from their use in colloquial discourses, which are less specialized. She uses the word routine broadly, including everyday communications in which greetings are routine. In the case of mathematical routine, the focus of Sfard's study, she uses it more in the creative sense of finding regularities or patterns, in a creative search for exploring mathematical discourse. These rules occur and are created or recreated during interactions between individuals.

Sfard and Kieran (2001, p. 70) state that: "if effective communication is generally difficult to achieve, in mathematics it is a real uphill struggle." The scarcity of perceptive mediation and the inherent polysemy in mathematical symbols can only be overcome by extreme concentration. Polysemy is present in words and symbols, which causes comprehension problems for students. Finally, for successful communication, it is crucial to maintain a well-defined attention. They conclude by stating that "one of the possibly most underrated skills that have to be fostered to enhance communication is the use of perceptual mediators, that is the ability to develop helpful attended foci" (Sfard & Kieran, 2001, p. 71).

3. RESEARCH METHODOLOGY

To answer the question – how does an experienced teacher introduce the concept of derivatives to 11th grade secondary school students in a public school in Lisbon? – the study adopted a case study approach, with data collected through classroom observations in a public secondary school. Teachers were generally reluctant to allow classroom observations, citing concerns about potential interference in the teaching-learning process. Among the several teachers consulted, only one agreed to participate in the study. The teacher works at a secondary school in Lisbon that has been in operation for 40 years, serving around 1,200 students, across three educational levels, regular secondary education and professional courses.

The observations took place in an 11^{th} grade classroom with 27 students, 20 of whom also attend physics and chemistry and descriptive geometry classes. It was a very heterogeneous class in terms of academic results. The focus of the observation was on the teacher and the dialogues with the students. The observation was carried out without intervention and interaction with the students. To ensure student anonymity, the teacher permitted only audio recordings, prohibiting video recordings. All classes were recorded also photos of the whiteboard were taken. The teacher's face was excluded from the images, which were included in the text.

In order to collect data, in the five classes that were observed, we followed the following script: 1) Record the approach that the teacher followed; 2) Identify the sequence of concepts presented, the use of words; 3) Record how she used the whiteboard – what did she write or draw?; 4) Record the main gestures used by the teacher: such as pointing with her finger at an object on the board (precisely or vaguely), using chalk as a pointer, using her hands to draw in the air, using the opening of her arms as an indication of large or small; 5) Identify how the interaction with the students occurred: did she ask questions or just use a monologue; did she ask and answer or wait for an answer.

After observing the five classes, each lasting approximately 90 minutes, a semi-structured interview was conducted with the teacher, with the aim of filling in any gaps that the observations may have left. After each observed class, the recordings were listened to and transcribed, and the main moments of dialogue in which the teacher introduced a new concept, explained an exercise or demonstrated a theorem were identified. These critical moments were called episodes. In the present text, I included six episodes.

For the analysis of the data, I followed Sfard (2008b) and her theory of *commocognition*, using the categories she selected for discourse analysis: 1) use of words; 2) visual mediators; 3) narrative; and, 4) routines. The category *routines* were subdivided into two subcategories: mathematical routines and classroom routines. Mathematical routines are understood as those defined by Sfard and classroom routines are those that are governed by sociomathematical norms, as described by Domingos (2005, p. 2). Sociomathematical norms are identified by "regularities in the patterns of social interaction that develop in the classroom". The analysis of the five classes observed is presented according to the proposed categories.

4. RESULTS

The main findings of this research, which will be detailed below, show that the teacher used a formalized approach, with some appeal to intuition. Her mathematical routine begins with a particular example; this is followed by a conjecture, demonstration and exercises. She uses the textbook as support in all classes. Traces of *commognition* occurred in two moments: in the final episode, when a student questioned the teacher about the possible existence of a derivative of the derivative, as well as in the assessment, in which the students presented positive results evidencing communication and understanding of the concept of derivative. The results obtained from the observations are presented according to the four categories described above.

4.1. Use of Words

The use of words shows which were the main concepts used in each class and reinforces Sfard's idea that repetition is the key to successful communication. The teacher never used the key words in a dubious sense or that caused confusion for the students, which shows the wisdom of the practice and the mastery of the knowledge to be taught, as Shulman (2014) defends. The presentation of the words used, which appear in Table 2, aims to show the sequence in which such words were presented and the repetitions in the classes observed.

Table 2. Use of Words by Teacher

1st class	Average rate of change, slope, tangent, average speed, limit, derivative, differential, approximation, incremental ratio, definition, demonstration
2nd class	Approximation, slope, derivative, limit, tangent line, normal line to the tangent, Ruffini's rule, differentiability, continuity, "modus ponnens", "modus tollens", definition, demonstration
3rd class	Lateral derivative, derivative function, slope, rules of derivation, slope, limit, definition, demonstration
4th class	Function, domain of existence, derivable function in a set, derivative of the sum, derivative of the product, derivative of the quotient, derivative of the composite function, limit, theorem, slope, differentiable, derivable, definition, demonstration.
5th class	Function, derivative, variation, slope, tangent line, maximum, positive, negative, plus infinity, minus infinity, increasing function, decreasing function, zero derivative, growth, minimum, strictly increasing, strictly decreasing, definition, demonstration

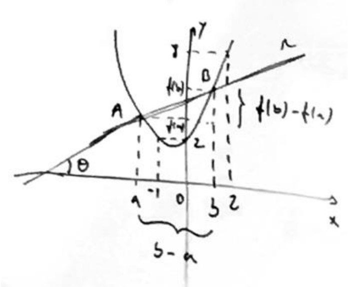
The construction of scientific discourse with the introduction of new words is constituted by the inculcation of new vocabulary through repetition. The formalized language of mathematics is supported by definitions, demonstrations, examples and counterexamples. It was a permanent discourse in all classes.

4.2. Visual Mediators

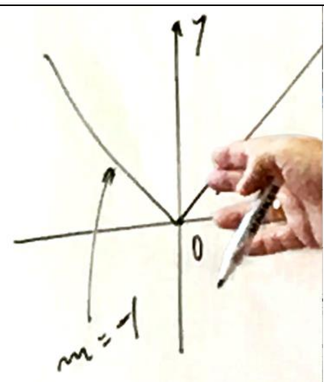
Mathematical discourses make massive use of symbolic artifacts, invented specifically for mathematical communication. In Table 3, I present some of these mediators that were used in the classes.

It is worth noting that the drawings were well-designed and clear, as were the different representations. The teacher's gestures were precise; she pointed directly to the mathematical object she was referring to. Sometimes, she physically touched the object written on the whiteboard with her finger to make it very clear what she was talking about. The teacher placed her hand on the board to point exactly to the mathematical object she was referring to, in this case, the graph of the modulus function.

Table 3. Examples of Visual Mediators



Graph: 1st Class



Point to: 2nd Class

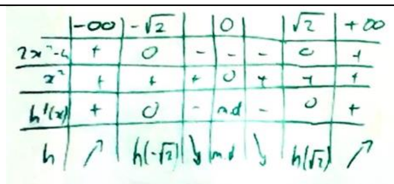


Table: 5th Class

Source: Photos of the whiteboard made by the researcher

4.3. Endorsed Narratives

In the first class, the mathematical narratives were presented through two definitions that the teacher explained on the board and orally, following the text: the definition of the average rate of variation and the derivative at a point. In the second class, the definition of "derivative function" written on the board coincides with the one in the textbook – "Given a real function of a real variable *f*, the derivative function of *f* is the domain function $D_{f'} = \{x \in D_f : f \text{ is differentiable at } x\}$ that each $x \in$

$D_{f'}$ corresponds to $f'(x)$." (Raposo & Gomes, 2023, p. 94). In her speech, the teacher emphasized that this was the expansion of a concept - that of the derivative of a function at a point - and associated this expansion with another already carried out with the continuous function at a point and a continuous function in a domain. She introduces the rules of derivation with examples.

The teacher used two rules of logic (modus ponens and modus tolens) when demonstrating and commenting on the theorem – the differentiability of a function at a point implies the continuity of the function at that point. The theorem was widely discussed with the students, showing examples and counterexamples of functions that, being continuous, are not derivable at a given point.

I consider, like Sfard (2008a), that these rules are a type of narrative, since they present themselves as a logical sequence of facts. "At more advanced levels of the colloquial discourse, and at any level of scholarly mathematical discourses, a narrative is as endorsable if it can be derived according to generally accepted rules from other endorsed narratives" (Sfard, 2008a, p. 223). The search for generalizations, which are expressed by theorems, is important in this and other classes observed. The teacher did not miss an opportunity to encourage students to master mathematical language: in the precision of definitions, notations and oral language. And, she corrected it whenever necessary.

4.4. Routines

The category routine was subdivided into two categories, which will be presented separately. Six episodes are presented that exemplify mathematical routines.

4.4.1. Math Routine

In the first class, the teacher makes use of an exercise in the textbook[1] in which was asked to calculate three average rates of variation for two nearby points and one generic point and established a dialogue with the students.

Although the teacher did not make a table with the values of the function in the vicinity of x=2, she showed this approximation by pointing to the graph; in doing so, she took advantage of this exercise and the fact that the students already knew the concept of limit to introduce the concept of derivative. I would also like to highlight the important connection between the new concept learned and reality that the teacher brought when presenting the average rate of variation as average speed. In a strategic pause, she informed that the new radars on the roads will measure not only the instantaneous speed but also the average speed to identify those violating the speed limit. Some students immediately expressed interest, wanting to know if the system was already working and on which roads. For Sfard (2008b), this connection with the environment should be a routine. By acting in this way, the effective teacher keeps the students attentive to his/her speech, ensuring the success of communication between teacher and student, which for Shulman (2014) means that the teacher had the competence of pedagogical knowledge of the content.

Episode 1. Limit of the average rate of variation

The teacher: Let's look at this a little more carefully. First, the authors of the textbook[2] put a 3, then 2,5 and, generally, x. When x approaches 2, what will happen?

[Exercise from the textbook: Let d(t)= $3t^2$+t be the distance, in meters, traveled in t seconds by a certain object. Calculate the average speed of the object in the intervals: a) [2,3], b) [2; 2.5], c) [2, x], (*with* x>2)]

Students: Don't Answer.

The teacher: It will be 6 m/s and then 14.5 m/s, and then 3x+7 m/s.

[After doing the calculations with the participation of the students].

[1] Raposo & Gomes, 2023, p. 86.

The teacher: If x approaches 2, how close will this expression be?

Student: 13

The teacher: If I take the limit, what does this correspond to?

Students: Don't answer

The teacher: In physics, what is this?

Students: Speed

The teacher: Instantaneous speed!

[The teacher adds]

The teacher: Instantaneous speed is exactly the limit of the average rate of change, which is called the derivative.

In this introductory lesson on the derivative of a function at a point, the teacher presented a mathematical object such as the average rate of change and slope of the secant line to a curve. This is an important connection. The connection of the algebraic expression with its graph or visual representation and notation is an important symbolic mediator. The students were encouraged to go through different representations for the concept of derivative.

The dialogue in *Episode 2* shows how the teacher constructed a mathematical narrative. She took advantage of the discussion on the geometric interpretation of the derivative to insist on the importance of mathematical demonstration.

Episode 2. Geometric interpretation of the derivative

The teacher: What is the derivative geometrically?

Student: It's the slope!

The teacher: It's the slope of the tangent line. If that's a line, what will happen at every point on the line?

[Teacher completes the student's answer]

Students: No answer

[The board says y=ax+b]

The teacher: What will be the slope at every point?

Student: It's the

The teacher: Give it! Let's calculate. Nothing like proof! So far it's just a conjecture.

We observed that, when the question is somewhat vague, such as "what will happen at all points on the line", the students remain silent, he teacher follows a very clear routine, which is outlined in Figure 1.

In *episode 3*, the teacher showed how, from an example in which the derivative of a related function is the coefficient of the independent variable, one can arrive at the rule for the derivative of a related function. The example used is the function f(x)= ax+b. Her initial comment was: Now, this is where the calculus begins!

Episode 3. Prove that the derivative of the function f(x)=ax+b is f'(x)=a

The teacher: Let's calculate f'(a). Now I think we should use the definition. What is that?

Students: Slope

The teacher: Slope of the tangent line

[teacher complements the answer of several students]

The teacher: Well, if that, that is a line [referring to the function f(x)]

The teacher: So what will happen *at all points*?

Students: a

The teacher: Our intuition tells us that it is so, but that is a conjecture. Nothing like a demonstration

The teacher: *What does that give?*

[calculates the limit by the definition and arrives at the value a]

Students: a

The teacher: So, the conjecture is proven.

arrives at the value a]

Students: a

The teacher: So, the conjecture is proven.

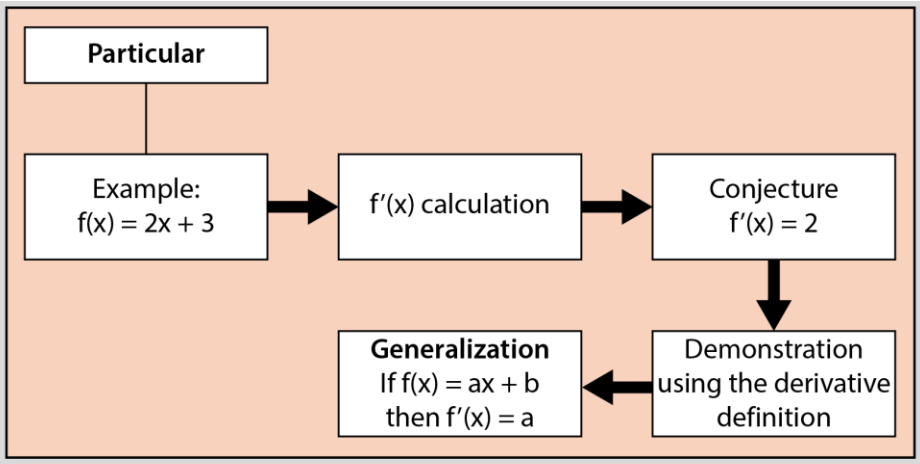


Figure 1. Scheme of Mathematical Routine

In this course of action, the teacher is seen giving students the opportunity to perceive the regularity that the derivative, at all points of any related function, will behave in the same way. An analysis of *Episode 3* clarifies that communication is a modeled activity; it clearly shows that communication patterns are dynamic structures and not invariable action schemes.

In *Episode 4*, the teacher established similarities and differences between the rule for the derivative of a product and the derivative of a quotient. The way she conducted the dialogue was an attempt to convince students to accept the rule for the derivative of the quotient as a particular case of the rule for the product, although there are differences in the results.

Episode 4. Derivative of the quotient

The teacher: Let f and g be two differentiable functions on a set D. If ∀a, a∈D and g(a)≠0, then $\left(\frac{f}{g}\right)' = \frac{f'(a).g(a)-g'(a)f(a)}{g^2(a)}$

[teacher speaks and writes on the whiteboard]

Student: What is D?

The teacher: It is the domain of each of the functions.

The teacher: Although the derivative of the quotient is very similar to that of the product, they have a difference.

The teacher: Look! What is the quotient if not a product. If I have f divided by g, how do I write this?

The teacher: f times 1/g

[teacher confirms]

The teacher: In practice this is differentiating a product. But the expression is slightly different.

Students: The minus sign and the denominator g^2.

This is an example of a routine in which the teacher, in an endorsed narrative, omits the proof. The teacher's justification that the same proof is very similar to the previous one could have been questioned by the listeners, however, they accepted the word of authority of the teacher and that of the textbook, which also omitted the proof.

The application of derivatives to the study of the monotony of functions occurred in the fifth class. In this class, the teacher only presented the statement of Lagrange's theorem, without a proof. No student demanded this proof. *Episode 5* shows how she explored the application of the theorem through an example. After the statement of the theorem written on the whiteboard, the teacher states that Lagrange's theorem allows us to prove this and moves on to the application of it to a particular function.

Episode 5. Application of the derivative

The teacher: We see here in the graph an intuitive analysis of the application of the derivative.

[teacher speaks and writes on the whiteboard]

The teacher: In practice, how do we do it?

Students: No answer

The teacher: This is how we study the variation of a function!

[Teacher writes on the whiteboard. Example: $f(x) = x^2 + 4x$]

The teacher: What do we do first?

Students Derivative

The teacher: What is the derivative of: $f(x) = x^2 + 4x$?

Student: 2x+4

The teacher: When the derivative is positive, the function is?

Students: Increasing

[Some students answer]

The teacher: When the derivative is negative, the function is?

Students: Decreasing [More than one student answers]

The teacher: Let's go.

[Writes 2x+4=0 on the board]

The teacher: For the analysis of the derivative, nothing is better than a sign board

[The teacher draws the sign board and completes it by asking the students questions].

Some of the exercises proposed in the textbook were presented by the students. In the fifth class, a volunteer student solved one of these exercises on the whiteboard, *Figure 4*. He was able to quickly apply the results to a quadratic function and analyze the growth and decrease of the function in R, using a table as a visual mediator.

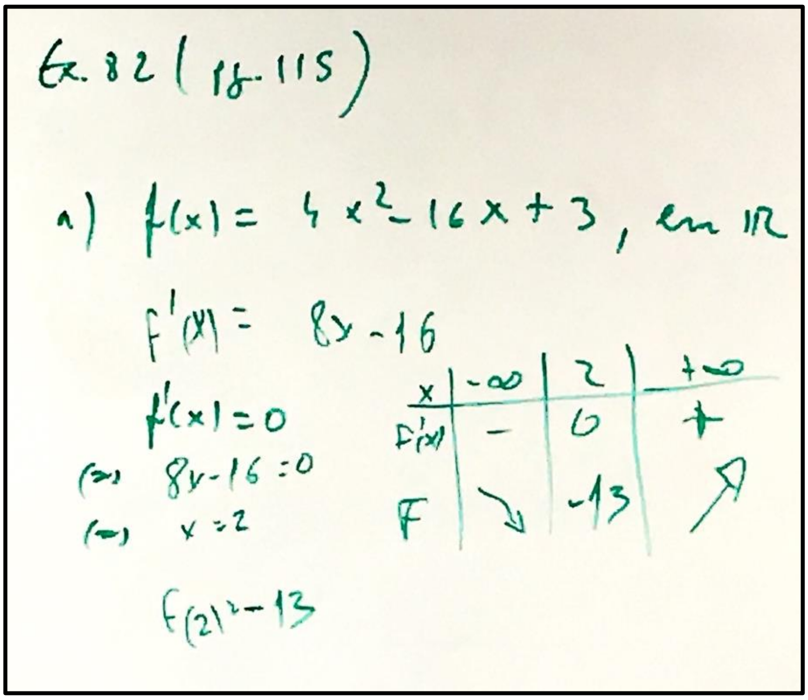


Figure 2. Exercise Solved on the Whiteboard

4.4.2. Classroom Routine

In broad terms, it can be said that the classroom routine followed the following steps: motivating example, definition, exercise and repetition of new vocabulary words. However, the observations allowed us to analyze what Domingos (2010) calls sociomathematical norms, which involve patterns of social interaction. For Yackel and Cobb (1996), sociomathematical norms "[...] reflect situations of interaction in the mathematics class, characterized by the normative understanding of what is considered mathematically different, mathematically sophisticated, mathematically effective or mathematically elegant" (Domingos, 2010, p. 202).

In all classes, the students always expected teacher to validate their answers. The authority of the teacher, who, for the students, held the knowledge, would legitimize their discourse. According to Domingos, "This rule can be called *conformation*, and its main role is to verify whether a given meaning is in accordance with what is considered mathematically valid" (Domingos, 2010, p. 205).

In all the classes observed, she began by correcting the exercises left as homework in the previous class. When presenting a new concept - for example, the geometric interpretation of the derivative at a point - she used the following sentence: "we have a tangent when the points are close, which is obvious". No student questioned the teacher about whether this was obvious; there seemed to be an agreement or, at least, if any student considered it not to be obvious, they did not say anything. After demonstrating a theorem, whose statement was copied on the board and demonstrated step by step, the teacher presented examples and counterexamples of functions that did or did not comply with the theorem. In the class on the rules of derivation, the students seemed excited to learn the rules, because they could practice more with them. The teacher warned that they would have this form available on the exam paper. She demonstrated the rule for the derivative of the sum of functions, the product rule, but not the quotient rule, saying that it was similar to the rule for the product. This demonstration did not appear in the textbook. Here, the lack of a demonstration, which was always considered essential, did not generate discussion. The students did not question the teacher, they tacitly accepted that it was so. However, when demonstrating the rule for the derivative of a product of two functions, the teacher was careful to do it step by step and show that, for this demonstration, it was necessary to know the theorem: "if a function is derivative at a point, then it is continuous at that point", to justify a limit crossing that, without the fact that the function **f** is derivative, might not exist". In the text, this passage is not justified as it should be, a gap that the teacher filled. In the exercises on the rules of derivation, the rule of symbolic comparison was widely used. This rule arises when it is necessary to manipulate more or less complex symbolic expressions.

In the 5th class, the students learned one of the applications of the derivative and seemed to have become much more motivated in relation to the subject. A brief comment by the teacher about the possibility of using graphing calculators to calculate derivatives prompted the following question from one of the students: "Why don't we make a calculator that is analytical?" The teacher's response, stating that such tools already existed, caused the student who had asked the question to smile broadly, an unmistakable expression of a mixture of joy and relief. A relief because it was not always necessary to solve derivatives by definition, that required so much algebraic effort. It is important to note that the teacher, in her practice, was very careful with words and notations on the board, correcting students on small writing mistakes.

A sign of the presence of *commognition* emerged at the end of the fifth and final class on derivatives, when a student expressed curiosity about derivatives.

Episode 6

Student: Is there a derivative of a derivative?

[students leaving the room]

The teacher: Yes. It is the second derivative.

[student smiles and seems pleased with his question and the teacher's answer]

The teacher: We will return to this subject next year.

[adds]

The student who asked this question always tried to interact in dialogues with the teacher, but he did not ask questions in class. This made me feel strange when he waited for the class to end and, discreetly, asked the question above, which had not been asked by any other student. His question gives signs of reflection or, as Sfard says, interpersonal communication or *commognition*.

5. CONCLUSION

The concept of derivative was constructed from the concept of average rate of variation, passing through the idea of approximation, limit and finally derivative at a point. The auxiliary means adopted were graphic visualizations, skillfully drawn on the board. The transition from the concept of derivative at a point to the concept of derivative in a set was presented as an expansion of the concept of derivative at a point. The approach to the concept of derivative was formalized, with some appeal to intuition. I agree with Sfard (2008a) when he says that this collective action between the teacher and the students generates *commognition*. I identified, in the effective communication between the interlocutors, that different statements evoked responses in tune with the speakers. However, we noted in the episodes described that the teacher did not encourage her students to have a more participatory dialogue, which would have given her the opportunity to listen more to the students and wait for them to come up with conjectures. My hypothesis is that the limited time of 5 classes to present so many new concepts and results prevented her from generating more mathematical discourses based on the students' reflection.

Although learning was not the focus of the investigation, an informal conversation with the teacher, after the evaluation on the derivatives, showed that the students' average was 13.9 points (out of a maximum of 20 points). Therefore, the learning result provided evidence of efficient teaching. As proposed by Shulman (1987), studies that record and analyze case studies, the actions of talented teachers can establish standards of practice in a specific area, in our case, in the teaching of mathematics. What the teacher said - and how she said it - made all the difference in the communication process in the classroom and showed that an efficient teacher can provide opportunities for learning.

Finally, I discuss one limitation of the study. Classroom observations were limited to a single teacher. If we had carried out at least the observations with two more teachers, we could have more comprehensive conclusions.

Acknowledgment. I thank professor António Domingos of Universidade Nova de Lisboa who appointed the school and professor for the field research.

Research Ethics. This study received ethics approval on 10 May 2024 from the secondary school Directorate (no number).

Data Availability Statement. All data can be obtained from the corresponding author.

Conflicts of Interest. I declare no conflicts of interest.

Funding. This research received no external funding.

REFERENCES

Artigue, M. (1998). L´évolution des problématiques en didactique del analyse. *Recherches en Didactique des Mathématiques, 18* (2), 231-261. https://revue-rdm.com/1998/l-evolution-des-problematiques-en/

Bressoud et al. (2016). *Teaching and Learning of Calculus.* ICME 2016. Springer Open.

Code, W., Piccolo, C., Kohler, D., & MacLean, M. (2014). Teaching methods comparison in a large calculus class. *ZDM Mathematics Education, 46*, 589–601. https://doi.org/10.1007/s11858-014-0582-2

Cornu, B. (1991). Limits. In: David Tall (ed) *Advanced mathematical thinking.* Kluwer, Dordrecht, 153–166.

Domingos, A. (2005). *Compreensão de Conceitos Matemáticos Avançados:* A Matemática no início do Superior. Tese (Doutorado em Teoria e Desenvolvimento de Currículo) - Universidade Nova de Lisboa Faculdade de Ciências e Tecnologia, Lisboa, 2005.

Domingos, A. M. D. (2010). As normas sociomatemáticas e a aprendizagem da matemática. *In XXI Seminário de Investigação em Educação Matemática,* 202-213.

Kidron, J. (2014). Calculus teaching and learning. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education.* p. 69-74. Springer.

Mkhastshwa, T. (2024). Best practices for teaching yhe concept of the derivative: Lessons from experienced calculus instructors. *Eurasia Journal of Mathematics, Science and Technology Education, 20*(4), 1-17. https://doi.org/10.29333/ejmste/14380

Nseanpa, C.; Gonzáles-Martín, A. (2022). Mathematics teacher's use of institutional resources to teach derivatives. The Case of Cameron. Conference: *Twelfth Congress of the European Society for Research in Mathematics Education* (CERME12), Italy. https://www.researchgate.net/publication/359919090_Mathematics_teachers'_use_of_institutional_resources_to_teach_derivatives_The_case_of_Cameroon

Park, J. (2015). Is the derivative a function? If so, how do we teach it? *Educational Studies in Mathematics*, *89*, 233-250. http://doi.org/10.1007/S10649-015-9601-7

Raposo, D., & Gomes, L. (2023). *Expoente 11*. Matemática A. Lisboa: Asa.

Sfard, A. (2008a). Introduction to thinking as communication. *TMME*, *5*(2&3), 429-435. https://www.researchgate.net/publication/357123327

Sfard, A. (2008b). *Thinking as communicating*: *Human development, the growth of discourses, and mathematizing.* Cambridge University Press.

Sfard, A., Kieran , C. (2001). Cognition as communication: rethinking learning-by-talking through multi-faceted analysis of students'mathematical interactions. *Mind, Culture, and Activity*, *8*(1), 42-76. https://doi.org/10.1207/S15327884MCA0801_04

Shulman, L. S. (2014). Conhecimento e ensino: fundamentos para a nova reforma. *Cadernoscenpec*, *4*(2), 196-229.

Shulman, L. S.; Shulman, J. (2016). Como e o que os professores aprendem: uma perspectiva em transformação. *Cadernoscen*pec**.** *São Paulo, 6*(1), 120-142.

Sierpinska, A (1985). Obstacles épistémologiques relatifs à la notion de limite [Epistemological obstacles relating to the concept of limit]. *Rech Didact Math, 6*(1), 5–67. https://revue-rdm.com/1985/obstacles-epistemologiques/

Tall, D. (ed) (1991) *Advanced mathematical thinking*. Kluwer, Dordrech.

Tall, D., Vinner, S. (1981) Concept image and concept definition in mathematics with particular reference to limit and continuity. *Educational Studies in Mathematics, 12*, 151–169. https://doi.org/10.1007/BF00305619

Yackel, E., e Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458-477. https://doi.org/10.2307/749877